J. Fluid Mech. (2006), *vol.* 554, *pp.* 125–146.
doi:10.1017/S0022112006009402 Printed in the United Kingdom

The stress system in a suspension of heavy particles: antisymmetric contribution

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(Received 24 February 2005 and in revised form 7 February 2006)

The nature of the stress in a suspension of equal homogeneous spheres all subject to the same force, such as weight, is considered; inertial effects are neglected. This study builds upon some of the well-known work devoted to this problem by the founder of the *Journal of Fluid Mechanics*, Professor George K. Batchelor. After developing a general theory, the antisymmetric part of the stress tensor is considered in detail. It is shown that, in addition to a term already found by Batchelor and characterized by an axial vector, the antisymmetric stress contains another term characterized by the curl of a polar vector. As a consequence, a suspension will possess, in addition to an axial vortex viscosity, a polar vortex viscosity. Appendix C presents a calculation of the hindrance function for rotation correct to the first order in the particle volume fraction.

1. Introduction

Among the many influential papers that George K. Batchelor published in his remarkable creation, the *Journal of Fluid Mechanics*, those devoted to the theory of suspensions marked the beginning of a new era for this discipline. In his first paper on this subject (Batchelor 1970), in open antithesis to the so-called axiomatic approach, he provided a physically insightful derivation of the nature of the stress tensor in a uniform suspension. Shortly thereafter, he presented a calculation of the first correction to the settling speed of a particle in a suspension elegantly side-stepping the convergence difficulties that had stumped earlier investigators (Batchelor 1972). It is fitting on this anniversary to pay homage to this work which has inspired two generations of researchers.

Batchelor focused on a spatially uniform system subject, in the first paper, to a spatially uniform bulk velocity gradient. For spherical force-free particles suspended in Stokes flow, he found that the stress consists of an isotropic contribution 'of no particular interest', an antisymmetric term proportional to the external couple acting on the particles, and a symmetric traceless component proportional to the rate of strain of the volumetric flux. This permits the definition of an effective viscosity for the suspension. The principal tool that he used in his analysis and, indeed, for all his work on suspensions, was ensemble averaging, which he carefully converted to volume averaging under the assumption of spatial uniformity.

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In this paper we re-examine the nature of the stress in a viscous suspension allowing for the presence of forces, such as weight, acting on the particles. We neglect inertial effects and consider only the case of equal homogeneous spherical particles. In the first part of the paper (§ § 2 and 3) we present a general theory relaxing the assumption of spatial uniformity made by Batchelor. In the second part of the paper (§ § 4–6) we focus in particular on the antisymmetric part of the stress tensor. We recover a term, characterized by an *axial* vector, already found by Batchelor, which essentially only exists when the particles are subject to external couples. However we also prove the existence of an additional term, characterized by a *polar* vector, as found by different arguments in earlier work (Marchioro, Tanksley & Prosperetti 2000; Marchioro *et al.* 2001; Prosperetti 2004). *Polar vectors* change sign upon a space inversion $\boldsymbol{x} \rightarrow -\boldsymbol{x}$; examples are the position vector, velocity, and others. An *axial vector*, such as vorticity, the moment of a force, and others, does not change sign upon a space inversion. The polar term vanishes only in the special situation of a spatially uniform distribution of particles, which seems to have been the only situation considered in the literature until recently. Because of the existence of two distinct contributions to the antisymmetric stress, a suspension will possess, in addition to an *axial* vortex viscosity (see e.g. Condiff & Dahler 1964; Brenner 1970, 1984), a *polar* vortex viscosity.

In the next section, we show how an expression for the mixture stress expressed solely in terms of the fluid stress can be derived; this step generalizes to the non-uniform case the procedure introduced by Batchelor (1970). Then (§ 3) we use some results of group theory to decompose this expression into an isotropic, a symmetric, and an antisymmetric component. In the second part of the paper we focus on the antisymmetric component, giving an analytic closure valid to second order in the volume fraction (§ 4), a numerical closure for the dense case (§ 5), and providing an intuitive physical interpretation of the results and of their implications (§ 6). A summary is given in § 7. Appendix C presents a calculation of the hindrance function for rotation correct to the first order in the particle volume fraction.

2. The total stress

While it is easy to write a formal expression for the mixture stress that involves the particle internal stress, such an expression is inconvenient as further progress requires consideration of the internal dynamics of the particles. A more useful description would involve only the stress in the fluid, with the particles considered as rigid entities without regard to the mechanics of their interior. Batchelor (1970) dealt with this issue by using the ergodicity property derived from his assumption of spatial homogeneity and a simple identity. In the present non-uniform case, his procedure must be suitably generalized, which is the purpose of this section.

We consider a suspension of N equal solid spheres of radius a in an incompressible viscous fluid. Under conditions of negligible inertia, the equation of motion at each point in the medium is

$$\nabla \cdot \boldsymbol{\sigma}_{F,P} - \nabla \psi_{F,P} = 0, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the stress tensor in the fluid (index F) or the particle material (index P) and ψ the deterministic potential of the respective body force. Upon taking an ensemble average, indicated by angle brackets, we have

$$\beta \langle \nabla \cdot \boldsymbol{\sigma}_P \rangle - \beta \nabla \psi_P = 0, \qquad (1-\beta)\langle \nabla \cdot \boldsymbol{\sigma}_F \rangle - (1-\beta)\nabla \psi_F = 0, \tag{2.2}$$

where β is the particle volume fraction.

Upon adding the equation for each phase and using the continuity of the normal stress at the particle surface, we have† (see e.g. Zhang & Prosperetti 1994, 1997)

$$\nabla \cdot [(1-\beta)\langle \boldsymbol{\sigma}_F \rangle + \beta \langle \boldsymbol{\sigma}_P \rangle] = (1-\beta)\nabla\psi_F + \beta\nabla\psi_P \tag{2.3}$$

where, with Batchelor's (1972) definition of ensemble averaging,

$$\beta\langle \boldsymbol{\sigma}_P \rangle = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(\mathscr{C}^N)\chi_P(\boldsymbol{x}|\mathscr{C}^N)\boldsymbol{\sigma}_P(\boldsymbol{x}|\mathscr{C}^N). \tag{2.4}$$

Here $P(\mathscr{C}^N)$ is the probability density of finding, in the realizations of the ensemble, the particle centers in the configuration $\mathscr{C}^N \equiv \boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^N$. The characteristic function of the particle phase, equal to 1 when the point $\boldsymbol{x}$ is inside a particle and 0 otherwise, is denoted by $\chi_P(\boldsymbol{x}|\mathscr{C}^N)$ and $\boldsymbol{\sigma}_P(\boldsymbol{x}|\mathscr{C}^N)$ is the stress at $\boldsymbol{x}$ when the N particles have the configuration $\mathscr{C}^N$. We explicitly note that we make no assumption of statistical spatial uniformity on P.

For equal spheres, the characteristic function can be written as $\chi_P = \sum_{\alpha=1}^{N} H(a - |\boldsymbol{x} - \boldsymbol{y}^\alpha|)$ so that

$$\beta(\boldsymbol{x}) = \int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\, n(\boldsymbol{x}+\boldsymbol{r}) = \left(1 + \frac{a^2}{10}\nabla^2 + \cdots\right)(nv) \tag{2.5}$$

where $v = \frac{4}{3}\pi a^3$ is the particle volume and it has been assumed that the spatial scale of variation of the local number density n is much greater than the particle radius a; n is defined by

$$n(\boldsymbol{x}) = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(\mathscr{C}^N)\left[\sum_{\alpha=1}^{N}\delta(\boldsymbol{x}-\boldsymbol{y}^\alpha)\right] = \frac{1}{(N-1)!}\int \mathrm{d}\boldsymbol{y}^2\cdots\mathrm{d}\boldsymbol{y}^N P(\boldsymbol{x}, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^N). \tag{2.6}$$

In the *continuum limit*, in which the particle radius is much smaller than the macroscopic scale L, $\beta = nv + O(a^2/L^2)$. We note for future reference that a generalization of the definition (2.6) to the average $\overline{g}$ of a generic quantity g pertaining to an entire particle is

$$n(\boldsymbol{x})\overline{g}(\boldsymbol{x}) = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(\mathscr{C}^N)\left[\sum_{\alpha=1}^{N} g^\alpha \delta(\boldsymbol{x}-\boldsymbol{y}^\alpha)\right]. \tag{2.7}$$

This *particle average* of g at a point $\boldsymbol{x}$ is therefore the average of the values of g taken over all the realizations of the ensemble for which there is a particle centred at $\boldsymbol{x}$.

In order to eliminate the stress in the particles from the exact expression (2.3), Batchelor (1970) wrote

$$\beta\langle \boldsymbol{\sigma}_P \rangle = \frac{1}{V}\sum_{\alpha=1}^{N}\int_{v^\alpha} \boldsymbol{\sigma}_P^\alpha \,\mathrm{d}V^\alpha = \frac{1}{V}\sum_\alpha \left[a\int_{S^\alpha}(\boldsymbol{\sigma}_P\cdot\boldsymbol{n})\boldsymbol{n}\,\mathrm{d}S - a\int_{V^\alpha}(\nabla\cdot\boldsymbol{\sigma}_P)\boldsymbol{n}\,\mathrm{d}V\right] \tag{2.8}$$

with V the volume of the system, S^α and V^α the surface and volume of particle α, and $\boldsymbol{n}$ the outward unit normal. He then used the continuity of normal stresses at the particle surface to replace $\boldsymbol{\sigma}_P\cdot\boldsymbol{n}$ by $\boldsymbol{\sigma}_F\cdot\boldsymbol{n}$ in the first integral and the particle

† When $\beta\langle\nabla\cdot\boldsymbol{\sigma}_P\rangle$ is expressed in terms of $\nabla\cdot(\beta\langle\boldsymbol{\sigma}_P\rangle)$, a term $\boldsymbol{\sigma}_P\cdot\boldsymbol{n}$ localized at the particle surface arises, which is an internal force for the suspension as a whole and is cancelled by an equal term derived from the same transformation operated upon $(1-\beta)\langle\nabla\cdot\boldsymbol{\sigma}_F\rangle$.

material momentum equation to relate the integrand in the last integral to the particle acceleration. In the second part of his paper, in order to derive specific expressions, he dropped the last term, neglecting inertia and considering force-free particles. In the general case of non-uniform suspensions, one cannot use ergodicity to calculate the ensemble average and it is necessary to proceed differently.

It is intuitively clear (and it is readily rigorously shown, see e.g. Prosperetti 1998, 2004) that one may write

$$\beta(\boldsymbol{x})\langle\boldsymbol{\sigma}_P\rangle(\boldsymbol{x}) = \int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\, n(\boldsymbol{x}-\boldsymbol{r})\langle\boldsymbol{\sigma}_P\rangle_1(\boldsymbol{x}|\boldsymbol{x}-\boldsymbol{r}), \tag{2.9}$$

where $\langle\boldsymbol{\sigma}_P\rangle_1(\boldsymbol{x}|\boldsymbol{x}-\boldsymbol{r})$ is the ensemble-average stress in the particle phase at $\boldsymbol{x}$ conditional on the presence of a particle centred at $\boldsymbol{x}-\boldsymbol{r}$. While $\langle\boldsymbol{\sigma}_P\rangle_1(\boldsymbol{x}|\boldsymbol{x}-\boldsymbol{r})$ varies rapidly with respect to the first variable, which identifies the position inside a particle, it varies slowly with respect to the second variable, which is the position of the particle centre. This permits us to carry out a Taylor series expansion to find (Zhang & Prosperetti 1994, 1997)

$$\beta\langle\boldsymbol{\sigma}_P\rangle(\boldsymbol{x}) = n(\boldsymbol{x})\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\langle\boldsymbol{\sigma}_P\rangle_1(\boldsymbol{x}+\boldsymbol{r}|\boldsymbol{x}) - \nabla_x\cdot\left[n(\boldsymbol{x})\int_{r\leqslant a}\mathrm{d}\boldsymbol{r}\,\boldsymbol{r}\langle\boldsymbol{\sigma}_P\rangle_1(\boldsymbol{x}+\boldsymbol{r}|\boldsymbol{x})\right] + \cdots. \tag{2.10}$$

In a homogeneous suspension the second and all higher terms vanish and this relation simply states that the average particle stress at $\boldsymbol{x}$ is the average of the stress over all the particles which contain $\boldsymbol{x}$ in their interior or on their surface. In general, this expression can be written in a more convenient way in terms of the particle average (2.7) by noting that, clearly (see e.g. Prosperetti 1998, 2004),

$$\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\langle\boldsymbol{\sigma}_P\rangle_1(\boldsymbol{x}+\boldsymbol{r}|\boldsymbol{x}) = \overline{\int_{r\leqslant a}\mathrm{d}\boldsymbol{r}\,\boldsymbol{\sigma}_P(\boldsymbol{x}+\boldsymbol{r}|\boldsymbol{x}, N-1)}, \tag{2.11}$$

where here and in the following the overline denotes the particle average as defined in (2.7). The left-hand side of (2.11) is the volume integral of the average stress while the right-hand side is the average of the integral of the stress. The use of (2.11) is implicit in the first equality of the relation (2.8) used by Batchelor (1970). This equation is simply a consequence of the commutativity of averaging and spatial integration, which is due to the linear nature of the latter operation. The second term of (2.10) can be written similarly so that

$$\beta\langle\boldsymbol{\sigma}_P\rangle(\boldsymbol{x}) = n(\boldsymbol{x})\overline{\int_{r\leqslant a}\mathrm{d}\boldsymbol{r}\,\boldsymbol{\sigma}_P(\boldsymbol{x}+\boldsymbol{r}|\boldsymbol{x}, N-1)} - \nabla_x\cdot\left[n(\boldsymbol{x})\overline{\int_{r\leqslant a}\mathrm{d}\boldsymbol{r}\,\boldsymbol{\sigma}_P(\boldsymbol{x}+\boldsymbol{r}|\boldsymbol{x}, N-1)\boldsymbol{r}}\right] + \cdots. \tag{2.12}$$

For the first term, we proceed as in the second step of (2.8) to find, upon using (2.1),

$$\overline{\int_{r\leqslant a}\mathrm{d}\boldsymbol{r}(\boldsymbol{\sigma}_P)_{ij}(\boldsymbol{x}+\boldsymbol{r}|\boldsymbol{x}, N-1)} = a\overline{\int_{r=a}\mathrm{d}S_r(\boldsymbol{\sigma}_F\cdot\boldsymbol{n})_i n_j} - \frac{\partial}{\partial x_i}\int \mathrm{d}\boldsymbol{r}\,\psi_P(\boldsymbol{x}+\boldsymbol{r})r_j. \tag{2.13}$$

For the second term in (2.12), we note that

$$\frac{\partial}{\partial x_k}\int \mathrm{d}\boldsymbol{r} r_k(\sigma_P)_{ij} = \frac{1}{2}\frac{\partial}{\partial x_k}\int \mathrm{d}\boldsymbol{r}[r_k(\sigma_P)_{ij} + r_j(\sigma_P)_{ik}] + \frac{1}{2}\frac{\partial}{\partial x_k}\int \mathrm{d}\boldsymbol{r}[r_k(\sigma_P)_{ij} - r_j(\sigma_P)_{ik}]. \quad (2.14)$$

The second contribution is antisymmetric in j and k and therefore vanishes upon taking the divergence over the index k indicated in (2.12) and the divergence over the index j to form the momentum equation (2.3) for the mixture, and can therefore can be dropped. The first contribution can be manipulated as before to find

$$\frac{1}{2}\int \mathrm{d}\boldsymbol{r}[r_k(\sigma_P)_{ij} + r_j(\sigma_P)_{ik}] = \frac{1}{2}a^2\int \mathrm{d}S_r n_k n_j(\boldsymbol{\sigma}_F\cdot\boldsymbol{n})_i - \frac{1}{2}\int \mathrm{d}\boldsymbol{r} r_k r_j \frac{\partial(\sigma_P)_{i\ell}}{\partial r_\ell}. \quad (2.15)$$

Similarly to (2.13), we have

$$\overline{\int \mathrm{d}\boldsymbol{r} r_k r_j \frac{\partial(\sigma_P)_{i\ell}}{\partial r_\ell}} = \int \mathrm{d}\boldsymbol{r} r_k r_j \frac{\partial \psi_P}{\partial r_i} = \frac{\partial}{\partial x_i}\int \mathrm{d}\boldsymbol{r} r_k r_j \psi_P. \quad (2.16)$$

Upon considering the two contributions (2.13) and (2.16) together, to the same order of accuracy as the terms retained in (2.5), we have

$$\nabla\cdot\left[n\nabla\int \mathrm{d}\boldsymbol{r}\psi_P\boldsymbol{r} - \frac{1}{2}\nabla\cdot\left(n\nabla\int \mathrm{d}\boldsymbol{r}\boldsymbol{r}\boldsymbol{r}\psi_P\right)\right] = (\beta - nv)\nabla\psi_P, \quad (2.17)$$

the proof of which rests on the assumption that ψ_P is harmonic as is usually the case. We now relate this result to the average force exerted by the particles on the fluid to obtain the final expression for the stress in the mixture.

For this purpose, we start by integrating the momentum equation (2.1) for the particle material over the volume of a generic particle centred at $\boldsymbol{y}$ and use the continuity of the normal stress at the particle surface to find

$$\int_{r=a} \mathrm{d}S_r\boldsymbol{\sigma}_F(\boldsymbol{y}+\boldsymbol{r}|\boldsymbol{y}, N-1)\cdot\boldsymbol{n} = \int_{r\leqslant a}\mathrm{d}\boldsymbol{r}\nabla_r\psi_P(\boldsymbol{y}+\boldsymbol{r}) = v\nabla_x\psi_P(\boldsymbol{y}), \quad (2.18)$$

where ∇_r denotes the gradient operator with respect to $\boldsymbol{r}$. The notation in the first integral indicates that the stress is evaluated on the surface of the particle centred at $\boldsymbol{y}$ while the position of the remaining $N-1$ particles is arbitrary. By using (2.6), (2.7), and (2.18), we may write the average hydrodynamic force on the particles as

$$\boldsymbol{T}(\boldsymbol{x}) \equiv \overline{\int_{r=a}\mathrm{d}S_r\boldsymbol{\sigma}_F\cdot\boldsymbol{n}} = v\nabla\psi_P(\boldsymbol{x}). \quad (2.19)$$

If inertia is neglected, the average external force per unit volume exerted on the particles must balance the fluid force and therefore it is given by

$$n(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x}) = -\frac{1}{v}\int_{r\leqslant a}\mathrm{d}\boldsymbol{r} n(\boldsymbol{x}+\boldsymbol{r})\boldsymbol{T}(\boldsymbol{x}+\boldsymbol{r}) \simeq -\left(1+\frac{a^2}{10}\nabla^2+\cdots\right)(n\boldsymbol{T}), \quad (2.20)$$

or

$$-n\boldsymbol{F} - \frac{a^2}{10}\nabla^2(n\boldsymbol{T}) = nv\nabla\psi_P. \quad (2.21)$$

Upon using (2.13), (2.15), (2.17), and (2.21), the total momentum equation (2.3) may be written as

$$\nabla\cdot\boldsymbol{\Sigma} = (1-\beta)\nabla\psi_F - n\boldsymbol{F}, \quad (2.22)$$

where we have introduced the mixture stress defined by

$$\boldsymbol{\Sigma} = (1-\beta)\langle\boldsymbol{\sigma}_F\rangle + an(\boldsymbol{x})\overline{\int_{r=a} \mathrm{d}S_r(\boldsymbol{\sigma}_F\cdot\boldsymbol{n})\boldsymbol{n}} - \frac{a^2}{2}\partial_k\left[n(\boldsymbol{x})\overline{\int_{r=a} \mathrm{d}S_r(\boldsymbol{\sigma}_F\cdot\boldsymbol{n})\boldsymbol{n}n_k}\right] + \frac{a^2}{10}\nabla(n\boldsymbol{T}) + \cdots. \quad (2.23)$$

The terms in the second line and the higher-order ones denoted by the dots reflect the fundamentally non-local nature of the theory due to the finite size of the particles. These terms vanish in the continuum limit but must be retained to accurately determine the effective properties by extrapolating numerical results to an infinite system size as shown in Zhang & Prosperetti (2006).

Equation (2.22) states that the total external force on the fluid and on the particles is balanced by $\nabla\cdot\boldsymbol{\Sigma}$; thus, we are justified in identifying $\boldsymbol{\Sigma}$ with the mixture stress. This representation of $\boldsymbol{\Sigma}$ is useful as any reference to the particle internal stress has now gone, which permits the use of a rigid-particle model. In this limit, it is easy to show that (Zhang & Prosperetti 1997)

$$(1-\beta)\langle(\sigma_F)_{ij}\rangle = -(1-\beta)\langle p_F\rangle\boldsymbol{I} + \mu[\nabla\boldsymbol{u}_m + (\nabla\boldsymbol{u}_m)^T] \quad (2.24)$$

where $\boldsymbol{I}$ the identity two-tensor, p_F is the fluid pressure, μ its viscosity, and

$$\boldsymbol{u}_m = \beta\overline{\boldsymbol{w}} + (1-\beta)\langle\boldsymbol{u}_F\rangle, \quad (2.25)$$

with $\overline{\boldsymbol{w}}$ the average translational velocity of the particles, is the volumetric flux, or bulk velocity, of the mixture.†

The derivation of (2.22) is presented in greater generality – but with an attendant greater complexity – in Prosperetti (2004) where, among others, (2.17) is proven to be exact at all orders. That paper however does not present the closure relations shown here. Tanksley & Prosperetti (2001) consider the Stokes flow case specifically, but again without showing explicit closure results.

3. Decomposition of the particle contributions to the stress

For brevity it is convenient to define the tensors

$$T_{ijk\ldots\ell} = \overline{\int_{r=a} \mathrm{d}S_r(\boldsymbol{\sigma}_F\cdot\boldsymbol{n})_i n_j n_k\cdots n_\ell}, \quad (3.1)$$

of which $\boldsymbol{T} \equiv \{T_i\}$ defined in (2.19) is a special case. Other tensors which arise in the analysis have the form

$$U_{k\ldots\ell} = \overline{\int_{r=a} \mathrm{d}S_r(\boldsymbol{n}\cdot\boldsymbol{\sigma}_F\cdot\boldsymbol{n})n_k\cdots n_\ell} = T_{jjk\cdots\ell}. \quad (3.2)$$

For brevity, we do not use a specific notation to identify the order of the tensors $\boldsymbol{T}$ and $\boldsymbol{U}$ that arise in the analysis; the order is implicit in the number of indices displayed. The first-order tensors $\{T_i\}$ and $\{U_i\}$ will also be written as $\boldsymbol{T}$ and $\boldsymbol{U}$ wherever convenient.

By a direct calculation based on Lamb's general solution of the Stokes flow past a sphere (Lamb 1932; Kim & Karrila 1991), it is shown in Prosperetti (2004) that the

† More precisely, the particle contribution to the volumetric flux should be written as $\beta\langle\boldsymbol{u}_P\rangle$, where $\langle\boldsymbol{u}_P\rangle(\boldsymbol{x})$ is the average velocity of the particle material at point $\boldsymbol{x}$. Approximating this quantity by $\overline{\boldsymbol{w}}$ introduces an error of order a/L, in which L is the macroscopic length scale (see e.g. equation 7.9 in Tanksley & Prosperetti 2001).

viscous part of the stress does not contribute to these latter tensors so that

$$U_{k\ldots\ell} = -\overline{\int_{r=a} \mathrm{d}S_r\, p_F n_k \cdots n_\ell}. \tag{3.3}$$

In terms of the $\boldsymbol{T}$, the previous expression (2.23) for the stress becomes

$$\Sigma_{ij} = (1-\beta)\langle(\sigma_F)_{ij}\rangle + naT_{ij} - \tfrac{1}{2}a^2\partial_k\left[n\left(T_{ijk} - \tfrac{1}{5}\delta_{jk}T_i\right) + \cdots\right]. \tag{3.4}$$

The first particle contribution T_{ij} can be decomposed as

$$T_{ij} = \hat{T}_{ij} + \tfrac{1}{2}(T_{ij} - T_{ji}) + \tfrac{1}{3}\delta_{ij}T_{\ell\ell} = \hat{T}_{ij} + \tfrac{1}{2}\epsilon_{ijn}\epsilon_{nk\ell}T_{k\ell} + \tfrac{1}{3}\delta_{ij}T_{\ell\ell}, \tag{3.5}$$

where $\hat{T}_{ij}$, which may be considered as defined by this relation, is seen to be the symmetric traceless part of T_{ij}, i.e. proportional to the average stresslet. From the definition (3.1) of T_{ij}, the second term of (3.5) is recognized as proportional to the average hydrodynamic couple acting on the particle. In the absence of inertia, the hydrodynamic couple must balance the applied external couple and therefore, if no external couples act on the particles, this term must vanish leaving only a symmetric contribution to the stress at this order. These results are the same as in Batchelor (1970) for the present case of spheres. The last term of the decomposition (3.5) gives an isotropic contribution to the stress and is given by

$$T_{\ell\ell} = \overline{\int_{r=a} \mathrm{d}S_r(\boldsymbol{n}\cdot\boldsymbol{\sigma}_F\cdot\boldsymbol{n})} = -\overline{\int_{r=a} \mathrm{d}S_r\, p_F} \equiv -4\pi a^2\langle\langle p_F\rangle\rangle \tag{3.6}$$

in which $\langle\langle p_F\rangle\rangle$ is defined as the particle average of the mean fluid pressure on the particle surface. It will be seen below that this term contributes to the particle pressure, as already pointed out by Brady (1993) and Jeffrey, Morris & Brady (1993).

T_{ij} is a reducible representation of the rotation group of order 2, and (3.5) corresponds to its decomposition into irreducible representations of order 2, 1, and 0 respectively. When T_{ijk} in the last term of (3.4), a reducible representation of order 3, is subjected to a similar analysis, one finds an irreducible fully symmetric 3-representation $\hat{T}_{ijk}$, two irreducible 2-representations, and three 1-representations; explicitly, following the procedure of Damour & Iyer (1991),

$$\begin{aligned} T_{ijk} = \hat{T}_{ijk} &+ \tfrac{1}{3}(T_{ijk} - T_{jik}) + \tfrac{1}{3}(T_{ijk} - T_{kji}) \\ &+ \tfrac{1}{15}[\delta_{ij}(T_k + 2U_k) + \delta_{jk}(T_i + 2U_i) + \delta_{ki}(T_j + 2U_j)]. \end{aligned} \tag{3.7}$$

Recall that it is not T_{ijk} itself that contributes to the stress in (2.12) but $\partial_k(nT_{ijk})$. Furthermore, what enters the dynamical equation is the divergence of the stress so that the contribution of dynamical significance is actually $\partial_j\partial_k(nT_{ijk})$. Because of this, it is readily verified that (3.7) can be written in the dynamically equivalent form

$$T_{ijk} - \tfrac{1}{5}\delta_{jk}T_i = \hat{T}_{ijk} + \tfrac{2}{3}(T_{ijk} - T_{jik}) + \tfrac{2}{15}[\delta_{ij}(T_k + 2U_k) - \delta_{jk}(T_i - U_i)]. \tag{3.8}$$

Furthermore, since the isotropic part of this term will ultimately contribute to the mixture pressure, it is justified to remove viscous effects from the term proportional to δ_{ij}. This objective can be achieved by adding to (3.8) the term $\frac{2}{15}[\delta_{ik}(T_j - U_j) - \delta_{ij}(T_k - U_k)]$, the double divergence of which vanishes. The result is then

$$\begin{aligned} T_{ijk} - \frac{1}{5}\delta_{jk}T_i &= \hat{T}_{ijk} + \tfrac{2}{3}(T_{ijk} - T_{jik}) + \tfrac{2}{15}[\delta_{ik}(T_j - U_j) - \delta_{jk}(T_i - U_i)] + \tfrac{2}{5}\delta_{ij}U_k \\ &= \hat{T}_{ijk} + \tfrac{2}{3}\epsilon_{ijn}\epsilon_{n\ell m}T_{\ell mk} + \tfrac{2}{15}\epsilon_{ijn}\epsilon_{nk\ell}(T_\ell - U_\ell) + \tfrac{2}{5}\delta_{ij}U_k. \end{aligned} \tag{3.9}$$

With this step, the isotropic part has been reduced to U_k which only depends on the fluid pressure, as remarked earlier after (3.2). As explained in Appendix A after (A 7), it is also desirable to symmetrize the second term with respect to the indices n and k:

$$\begin{aligned}\epsilon_{n\ell m}T_{\ell mk} &= \tfrac{1}{2}\left(\epsilon_{n\ell m}T_{\ell mk}+\epsilon_{k\ell m}T_{\ell mn}\right)+\tfrac{1}{2}\left(\epsilon_{n\ell m}T_{\ell mk}-\epsilon_{k\ell m}T_{\ell mn}\right)\\ &= \tfrac{1}{2}\left(\epsilon_{n\ell m}T_{\ell mk}+\epsilon_{k\ell m}T_{\ell mn}\right)+\tfrac{1}{2}\epsilon_{kn\ell}\left(T_\ell-U_\ell\right)\end{aligned} \tag{3.10}$$

where, in the last step, we used an identity which is readily proven from the definition of the tensors $\boldsymbol{T}$. With this step, we finally have

$$T_{ijk}-\tfrac{1}{5}\delta_{jk}T_i=\hat{T}_{ijk}+\tfrac{1}{3}\epsilon_{ijn}\left(\epsilon_{n\ell m}T_{\ell mk}+\epsilon_{k\ell m}T_{\ell mn}\right)-\tfrac{1}{5}\epsilon_{ijn}\epsilon_{nk\ell}\left(T_\ell-U_\ell\right)+\tfrac{2}{5}\delta_{ij}U_k. \tag{3.11}$$

Upon substituting (3.5) and (3.11) into (3.4) and suitably arranging the result, the mixture stress (3.4) may then be written in the form

$$\boldsymbol{\Sigma}=-p_m\boldsymbol{I}+\boldsymbol{\Sigma}_s+\boldsymbol{\epsilon}\cdot(\boldsymbol{R}-\nabla\times\boldsymbol{V}) \tag{3.12}$$

where $\boldsymbol{\epsilon}$ is the alternating tensor. The mixture pressure p_m is found to be

$$p_m=(1-\beta)\langle p_F\rangle+\beta\langle\langle p_F\rangle\rangle+\frac{a^2}{5}\partial_k\left(nU_k\right), \tag{3.13}$$

where we have written β in place of nv and, here and in the following, we drop explicit indication of higher-order terms in a/L. The symmetric part of the stress is given by

$$\Sigma_{s,ij}=\mu\left[\nabla\boldsymbol{u}_m+(\nabla\boldsymbol{u}_m)^T\right]_{ij}+an\hat{T}_{ij}-\frac{a^2}{2}\partial_k(n\hat{T}_{ijk}). \tag{3.14}$$

The second term $an\hat{T}_{ij}$, proportional to the stresslet, is responsible for the effective viscosity of the suspension. In the non-uniform case, it also introduces additional terms described in Zhang & Prosperetti (2006). The term $\hat{T}_{ijk}$ is shown in Appendix A to vanish in the uniform case and, more generally, to give contributions of order a^2/L^2 which vanish in the continuum limit.

The axial vector $\boldsymbol{R}$ in (3.12) is Batchelor's antisymmetric part of the stress augmented by an additional term:

$$\begin{aligned}R_i &= \frac{a}{2}\epsilon_{ik\ell}nT_{k\ell}-\frac{a^2}{6}\partial_k\left[n\left(\epsilon_{i\ell m}T_{\ell mk}+\epsilon_{k\ell m}T_{\ell mi}\right)\right]\\ &= \frac{1}{2}\overline{\int_{r=a}\mathrm{d}S_r\left[(\boldsymbol{\sigma}\cdot\boldsymbol{n})\times\boldsymbol{r}\right]_i}-\frac{1}{6}\partial_k\overline{\int_{r=a}\mathrm{d}S_r\left\{\left[(\boldsymbol{\sigma}\cdot\boldsymbol{n})\times\boldsymbol{r}\right]_i r_k+\left[(\boldsymbol{\sigma}\cdot\boldsymbol{n})\times\boldsymbol{r}\right]_k r_i\right\}}\end{aligned} \tag{3.15}$$

It is shown in Appendix A that the second integral vanishes up to terms of order a/L.

The polar vector $\boldsymbol{V}$ in (3.12) represents a new contribution to the antisymmetric part of the stress, which vanishes in the cases treated by Batchelor. To the present order of accuracy of the expansion (2.10) of the particle contribution to the stress it is given by

$$\boldsymbol{V}=\frac{a^2}{10}n\left(\boldsymbol{U}-\boldsymbol{T}\right)=-\frac{a^2}{10}n\overline{\int_{r=a}\mathrm{d}S_r\left(\boldsymbol{I}-\boldsymbol{nn}\right)\cdot\left(\boldsymbol{\sigma}_F\cdot\boldsymbol{n}\right)}, \tag{3.16}$$

in which $\boldsymbol{U}\equiv\{U_i\}$. This quantity is therefore proportional to the particle average of the tangential traction integrated over the particle surface. As such, it only depends on the viscous part of the stress. Since, in Stokes flow, the viscous and pressure stresses usually give contributions of comparable order to the hydrodynamic force, the integral in $\boldsymbol{V}$ may be expected to be of the order of magnitude of the drag. If the drag is estimated to be of the order of μa times the relative velocity, $\boldsymbol{V}$ is then expected to be of the order of $\beta\mu$ times the relative velocity.

As shown in Tanksley & Prosperetti (2001), in the present Stokes case the development leading from (2.3) to (2.22) facilitates but is not essential for the identification of this new contribution to the stress. A similar result also holds at finite Reynolds numbers, as shown in Prosperetti (2004).

4. Dilute limit

The previous results contain some terms that vanish identically in the situations studied by previous authors. It is interesting to investigate their nature and to consider their possible importance in more general situations. We start by a dilute-limit analysis, and present some numerical results for finite volume fractions in the next section.

The Stokes flow field in the neighbourhood of a particle can be represented as the sum of a part $(\boldsymbol{u}^r, p^r)$ and a part $(\boldsymbol{u}^s, p^s)$ regular and singular, respectively, at the particle centre. Explicit expressions for these quantities can be written down by means of Lamb's general solution of the Stokes equations summarized in Appendix A (Lamb 1932; see also Kim & Karrila 1991). The regular and singular components are connected to each other by the no-slip condition at the surface of the particle, assumed to translate with a velocity $\boldsymbol{w}$ and rotate with an angular velocity $\boldsymbol{\Omega}$. By using this representation and (A 5) and (A 6) of Appendix A it can be shown that

$$\boldsymbol{V} = \tfrac{3}{10}\beta\mu\left(\overline{\boldsymbol{w}} - \overline{\boldsymbol{u}^r} - \tfrac{1}{2}a^2\overline{\nabla^2\boldsymbol{u}^r}\right). \tag{4.1}$$

We consider a locally uniform sedimenting suspension of equal particles, all subject to the same force $\boldsymbol{f}$. The mean sedimentation velocity with respect to the bulk volumetric flux is related to the velocity $\boldsymbol{W}^0$ of an isolated particle subject to the same force by

$$\boldsymbol{W}^0 = \frac{\boldsymbol{f}}{6\pi\mu a} = \frac{1}{H(\beta)}\left(\overline{\boldsymbol{w}} - \boldsymbol{u}_m\right) \tag{4.2}$$

where H is the hindrance function for sedimentation for which Batchelor (1972) found

$$H = 1 - 6.55\beta + o(\beta). \tag{4.3}$$

On the other hand, from Faxén's theorem,

$$\boldsymbol{W}^0 = \overline{\boldsymbol{w}} - \overline{\boldsymbol{u}^r} - \frac{a^2}{6}\overline{\nabla^2\boldsymbol{u}^r} \tag{4.4}$$

so that

$$\boldsymbol{V} = \tfrac{3}{10}\beta\mu\left(\boldsymbol{W}^0 - \tfrac{1}{3}a^2\overline{\nabla^2\boldsymbol{u}^r}\right). \tag{4.5}$$

By using the approach of Batchelor's (1972) paper and some of his results, it is shown in Appendix B that

$$a^2\overline{\nabla^2\boldsymbol{u}^r} \simeq \beta\zeta(\beta)\boldsymbol{W}^0, \qquad \zeta = 3.537 + o(1) \tag{4.6}$$

so that

$$\begin{aligned}\boldsymbol{V} &= \tfrac{3}{10}\beta\left(1 - \tfrac{1}{3}\beta\zeta\right)\mu\boldsymbol{W}^0 = \frac{\beta(3-\beta\zeta)}{10H}\mu\left(\overline{\boldsymbol{w}} - \boldsymbol{u}_m\right)\\ &\simeq \tfrac{3}{10}\beta(1 + 5.37\beta)\,\mu\left(\overline{\boldsymbol{w}} - \boldsymbol{u}_m\right).\end{aligned} \tag{4.7}$$

It is shown in the next section that ζ is very nearly constant at least up to $\beta = 45\,\%$.

For the axial term of Batchelor's antisymmetric stress (3.15), we find

$$\frac{a}{2}\epsilon_{ik\ell} n T_{k\ell} = 3\mu\beta\left(\overline{\boldsymbol{\Omega}} - \tfrac{1}{2}\overline{\nabla \times \boldsymbol{u}^r}\right)_i = 3\mu\frac{\beta}{R(\beta)}\left(\overline{\boldsymbol{\Omega}} - \tfrac{1}{2}\nabla \times \boldsymbol{u}_m\right)_i, \tag{4.8}$$

in which $R(\beta)$ is the hindrance function for rotation. By again using the approach of Batchelor's (1972) paper, it is shown in Appendix C that

$$R \simeq 1 - 1.5\beta + o(\beta). \tag{4.9}$$

As shown in Appendix A, the second term of Batchelor's antisymmetric stress (3.15) vanishes in the continuum limit so that, to second order in β,

$$\boldsymbol{R} \simeq 3\beta\mu\,(1 - 1.5\beta)\left(\overline{\boldsymbol{\Omega}} - \tfrac{1}{2}\nabla \times \boldsymbol{u}_m\right). \tag{4.10}$$

The hindrance function for rotation was calculated numerically for finite volume fractions in Marchioro *et al.* (2000) and the results were fitted as

$$R = (1-\beta)^{c_1 - c_2\beta} \qquad \text{with} \qquad c_1 \simeq 1.50, c_2 \simeq 0.41, \tag{4.11}$$

which agrees with (4.9) for small β.

The last contribution to the mixture pressure (3.13) is found to be

$$\begin{aligned} \frac{a^2}{5}\nabla\cdot(n\boldsymbol{U}) &= \nabla\cdot\left(-\frac{a}{5}n\overline{\int_{r=a} \mathrm{d}S_r\, p_F \boldsymbol{r}}\right) = \nabla\cdot\left[-\frac{3}{10}\mu\beta\left(\boldsymbol{W}^0 + \frac{2}{3}a^2\overline{\nabla^2\boldsymbol{u}^r}\right)\right] \\ &= \nabla\cdot\left[-\frac{3}{10}\frac{\beta}{H}\left(1 + \frac{2}{3}\beta\zeta\right)\mu\,(\overline{\boldsymbol{w}} - \boldsymbol{u}_m)\right] \end{aligned} \tag{4.12}$$

and therefore, with

$$P = (1-\beta)\langle p_F\rangle + \beta\langle\langle p_F\rangle\rangle, \tag{4.13}$$

we have the general result

$$p_m = P - \nabla\cdot\left[\frac{3}{10}\frac{\beta}{H}\left(1 + \frac{2}{3}\beta\zeta\right)\mu\,(\overline{\boldsymbol{w}} - \boldsymbol{u}_m)\right] \tag{4.14}$$

which, in the dilute limit, reduces to

$$p_m = P - \nabla\cdot\left[\frac{3}{10}\beta\,(1 + 8.91\beta)\,\mu\,(\overline{\boldsymbol{w}} - \boldsymbol{u}_m)\right]. \tag{4.15}$$

The first term in the symmetric stress (3.14) is found to have the expression

$$a\hat{T}_{ij} = \tfrac{5}{2}\upsilon\mu\overline{\left(1 + \tfrac{1}{10}a^2\nabla^2\right)\left(\partial_i u_j^r + \partial_j u_i^r\right)}. \tag{4.16}$$

To first order in the volume fraction, the regular part of the velocity field can be identified with the average field and we therefore find from the first two terms of (3.14)

$$\boldsymbol{\Sigma}_s = \mu\left(1 + \tfrac{5}{2}\beta\right)\left[\nabla\boldsymbol{u}_m + (\nabla\boldsymbol{u}_m)^T\right] \tag{4.17}$$

where higher-order derivatives have been dropped as being of higher order in a/L. This is of course Einstein's result for the effective viscosity of a dilute suspension. The extension to the next order in β for a uniform suspension of force-free particles was given by Batchelor & Green (1972) in another very well-known paper. In the non-uniform case with particles subject to external forces, additional terms arise which are studied in detail in Zhang & Prosperetti (2006).

5. The polar contribution to the antisymmetric stress at finite volume fraction

In the standard calculation of the effective viscosity of a concentrated suspension one considers a uniform system subject to a uniform shear even though, strictly speaking, taking the divergence of the result to form the momentum equation would result in a zero contribution (see e.g. Batchelor & Green 1972; Phillips, Brady & Bossis 1988; Mo & Sangani 1994; Heyes & Sigurgeirsson 2004). Implicitly, the procedure rests on an assumed separation of scales such that the characteristic macroscopic length scale over which non-uniformities are dynamically significant is larger than the local length scale used to deduce the effective property. In order to calculate the new contribution $\boldsymbol{V}$ to the antisymmetric stress at finite particle concentrations we follow a similar approach and consider a uniform system, even though ultimately $\boldsymbol{V}$ enters the momentum equation (2.22) as $-\nabla \times \nabla \times \boldsymbol{V}$.†

It was shown in the previous section that, in the dilute limit, $\boldsymbol{V}$ is proportional to the normalized force $\boldsymbol{W}^0$. That a similar proportionality should hold also at finite volume fractions is a consequence of the fact that, for a uniform suspension, no vector other than $\boldsymbol{W}^0$ is available to formulate a closure relation for $\boldsymbol{V}$. We now calculate numerically the coefficient of proportionality or, equivalently, the coefficient $\zeta(\beta)$ introduced in (4.6).

For this purpose we carry out numerical simulations of the response of particle assemblies to a force, equal for all the particles, and take an ensemble average of the results. Each ensemble is statistically uniform and consists of realizations obtained by randomly arranging N particles of equal radius a in a cubic domain of side L with periodicity boundary conditions. In principle, each different flow situation will have a different particle probability distribution. In order to obtain a result of reasonably broad applicability, we are forced to make a single specific choice for this probability distribution. The hard-sphere distribution seems the most natural choice.

Since we are interested in the limit of large systems, for each volume fraction we constructed between 40 and 70 different ensembles, increasing the particle number and the box size according to the relation

$$\frac{a}{L} = \left(\frac{3}{4\pi N}\beta\right)^{1/3}. \tag{5.1}$$

With a maximum of 160 particles, this relation gives $a/L = 0.024$ and 0.088 for $\beta = 1\,\%$ and 45 %, respectively. For each particle number and cell size, our ensembles consist of a large number of particle configurations: 1024 for N between 17 and 79 512 for N between 80 and 150, and 256 for $N = 160$. For each realization, the force acting on the particles was taken in the three mutually orthogonal directions parallel to the sides of the cell, thus effectively tripling the number of realizations for each ensemble.

The numerical method, which is the same as described in detail in earlier papers (Marchioro *et al.* 2000, 2001; Ichiki & Prosperetti 2004), is based on a multipole

† Note that short-scale non-uniformities would add differentiated terms to the result found here, with the consequence that very high-order derivatives would appear in the momentum equation. This could alter the mathematical structure of the final equation system and necessitate new, non-standard boundary conditions.

β (%)	ζ	β (%)	ζ
1	3.56	20	3.53
2	3.55	25	3.50
3	3.53	30	3.46
4	3.54	35	3.94
5	3.55	40	3.37
10	3.54	45	3.36
15	3.53		

TABLE 1. Numerically computed ζ defined in (4.6) vs. particle volume fraction β.

expansion as developed by Mo & Sangani (1994), whose code formed the basis of the one that we have used. Five multipoles were retained for the majority of the calculations. We retained six for a limited number of high-concentration cases, finding differences of the order of 1–2 % for both the hindrance function and the coefficient ζ. As shown in (A 5), the quantity of interest, $\nabla^2 \boldsymbol{u}^r$, is directly proportional to one of the multipoles.

Figure 1 shows typical examples of the dependence of ζ upon a/L for β between 1 % and 45 %. In this figure, each point is the result of averaging over the ensemble corresponding to that value of a/L. The validity of the relation (4.6) is confirmed by the fact that the components of $\overline{\nabla^2 \boldsymbol{u}^r}$ not parallel to $\boldsymbol{W}^0$ are found to be orders of magnitude smaller than the parallel one.

The numerical results are smooth and, in order to extrapolate to $a/L = 0$, we fitted them by least squares with both a linear and a quadratic fit, finding very small differences. The results of this extrapolation are shown as a function of β in figure 2 and given in numerical form in table 1. The dashed line shows the dilute limit value $\zeta \simeq 3.537$ given earlier in (4.6) and the solid line is a fit of the numerical results of the form

$$\zeta = 3.54(1 - 0.271\beta^2). \tag{5.2}$$

6. Discussion

In the previous sections we have presented a formal derivation of the antisymmetric stress. It is useful to supplement it by an intuitive physical argument.

Consider a mixture volume element such as the one depicted in figure 3. On average, the surface of the volume element will cut across some particles such as A and B. As shown in the right part of the figure, due to the neglect of inertia, the external force $\boldsymbol{F}$ applied to particle A will be balanced by the sum of hydrodynamic forces $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ arising from the integration of the fluid stress on the portions of the surface of A inside and outside the volume element, respectively. (For simplicity, these forces are drawn parallel to the surface of the volume element.) The action of the forces $\boldsymbol{F}$ and $\boldsymbol{f}_2$ on the particle material inside the volume element can be equivalently represented by a force $\boldsymbol{F} - \boldsymbol{f}_2$ and by a couple $\boldsymbol{c}$ acting at the intersection of a particle with the surface of the volume element. A similar argument applies to a particle such as B, for which the force will be $-\boldsymbol{f}_2'$ (and, therefore, opposite to $\boldsymbol{F} - \boldsymbol{f}_2$), while the couple will be in the same direction.† The sum of terms similar to $\boldsymbol{F} - \boldsymbol{f}_2$ and $-\boldsymbol{f}_2'$ for all the particles cut

† This difference in signs explains why the $\boldsymbol{V}$ contribution comes in at $O(\beta)$, while that of the tensors E_Δ and E_∇ introduced in Zhang & Prosperetti (2006) starts at $O(\beta^2)$.

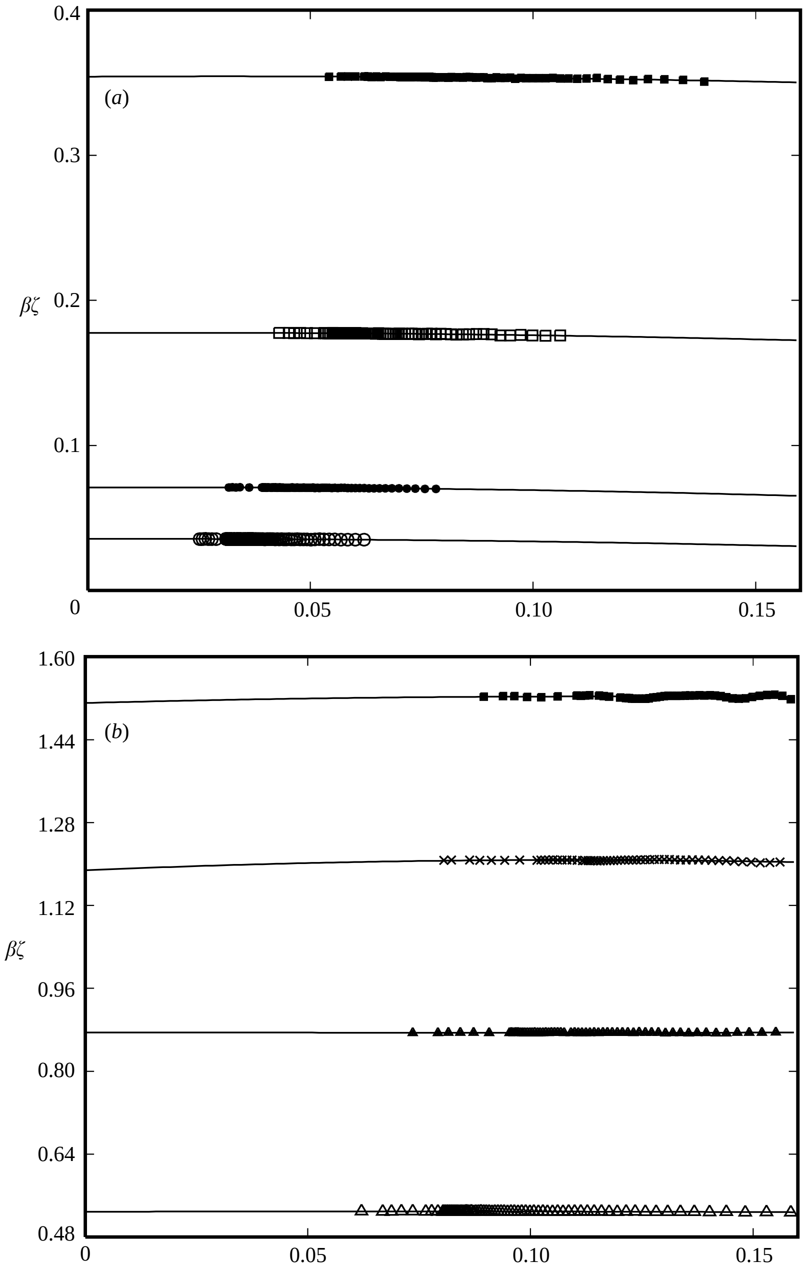

FIGURE 1. Numerically computed values of the coefficient ζ defined in (4.6) vs. the size L of the computational cell normalized by the particle radius. Each point is the result of averaging over a large ensemble of configurations. The lines are, in ascending order, for particle volume fractions of (*a*) 1 %, 2 %, 5 %, and 10 % and (*b*) 15 %, 25 %, 35 % and 45 %.

across by the surface of the volume element contributes to the symmetric stress, while the couple $\boldsymbol{c}$ contributes to the antisymmetric stress term $\boldsymbol{V}$. It is evident that this argument does not depend on whether external couples act on the particles or not. If they do, the antisymmetric stress will be augmented by the contribution $\boldsymbol{R}$. The

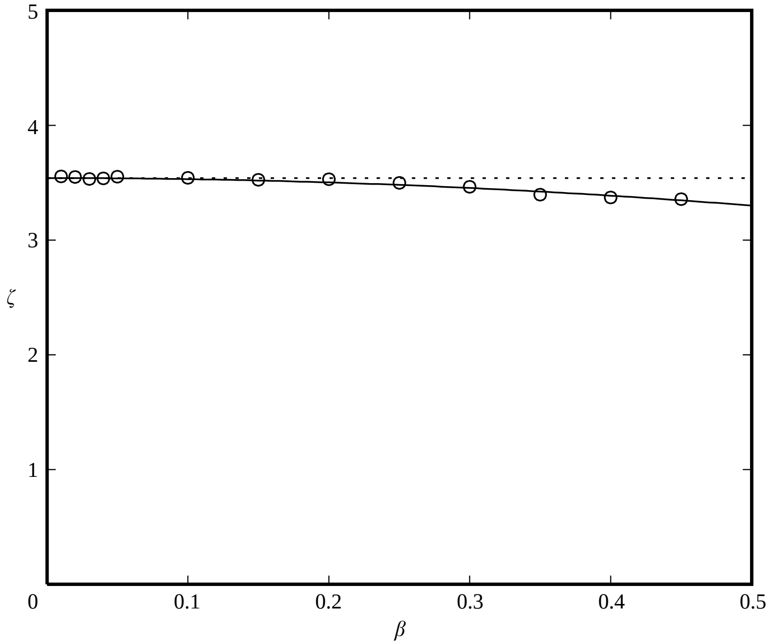

FIGURE 2. Dependence of the coefficient ζ defined in (4.6) on the particle volume fraction β. The symbols show the result of the large-system extrapolation to $a/L \to 0$, the dashed line is the dilute-limit result, and the solid line the fit (5.2).

crucial role played by particles cut across by the surface of the volume element (which must exist in both an ensemble average and in a properly justified volume average treatment) is appreciated well in the multiphase flow literature; see e.g. Nigmatulin (1979), Prosperetti & Jones (1984), and Almog & Brenner (1999).

The couple $\boldsymbol{c}$ will be proportional to the force $\boldsymbol{F}$ which, on average, will be $\mu(\overline{\boldsymbol{w}} - \boldsymbol{u}_m)/H$. If both $\boldsymbol{F}$ and the particle volume fraction are spatially uniform, the net effect integrated over the surface of the control volume will add up to zero, but this conclusion will not hold if either one, or both, of these conditions is not satisfied.

It is known from the standard theory of structured continua (see e.g. Aris 1962; Dahler & Scriven 1963; Condiff & Dahler 1964) that the pseudo-vector $-\epsilon_{ijk}\Sigma_{jk}$ of the antisymmetric component of the stress tensor, which in our case equals $-2(\boldsymbol{R}-\nabla\times\boldsymbol{V})$, is a source for the *intrinsic* angular momentum, namely the difference between the total angular momentum and the moment of the linear momentum. This difference is often termed the *spin field* as it has been traditionally regarded as accounting for rotating particles. Our analysis shows that this is actually an over-simplification as, due to the $\boldsymbol{V}$ term, the pseudo-vector does not vanish even when the particles are subject to no external couples and $\boldsymbol{R} = 0$. Thus it must be concluded that, in general, the suspension possesses an intrinsic angular momentum due to non-uniformities of the particle distribution and of the forces acting on them. In other words, under the action of $\nabla \times \boldsymbol{V}$, a differential rotation develops inside a volume element rotating with angular velocity $\frac{1}{2}\nabla \times \boldsymbol{u}_m$. Though a microscale effect, our results imply that this feature has macroscale consequences. This micro/macro coupling is mediated by the 'slip' velocity field $\overline{\boldsymbol{w}} - \boldsymbol{u}_m$ in the presence of spatial non-uniformities.

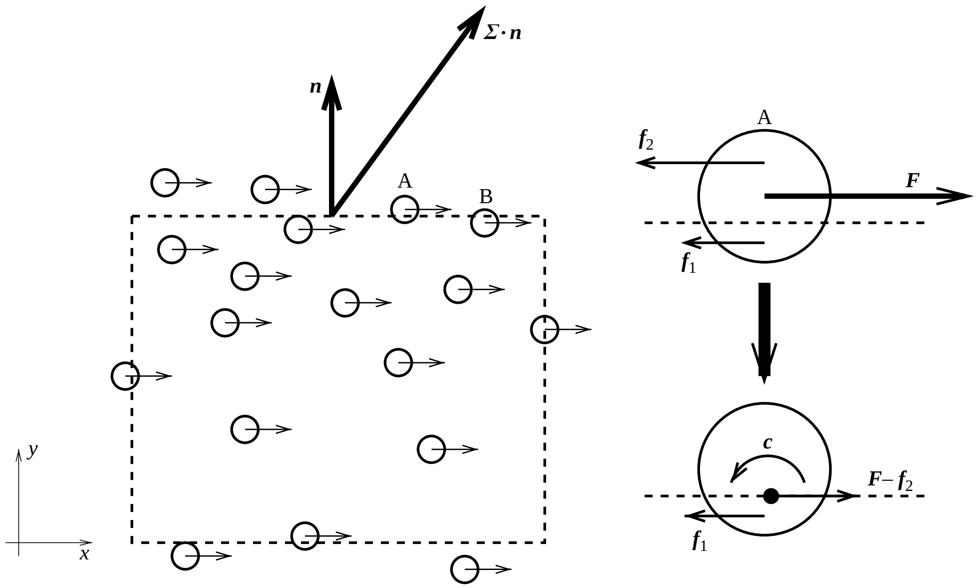

FIGURE 3. An intuitive explanation of the origin of the new antisymmetric contribution described in §§4 and 5. The external force $\boldsymbol{F}$ applied to particle A is balanced by the sum of hydrodynamic forces $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ arising from the integration of the fluid stress on the portions of the surface of A inside and outside the volume element. (For simplicity, these forces are drawn parallel to the surface of the volume element.) The action of the forces $\boldsymbol{F}$ and $\boldsymbol{f}_2$ on the particle material inside the volume element is equivalent to a force $\boldsymbol{F} - \boldsymbol{f}_2$ and a couple $\boldsymbol{c}$ acting at the intersection of the volume element with the particle. The sum over the couples $\boldsymbol{c}$ for all the particles intersected by the boundary of the volume element gives rise to an antisymmetric stress.

7. Summary and conclusions

In this paper we have re-examined the stress in a uniform suspension following the lead of Batchelor's famous 1970 paper. Unlike him, we have limited our considerations to spherical particles, but we have allowed the particles to be subjected to forces and couples. We have shown that, in the absence of inertial effects, the stress tensor has the form given in (3.12), namely

$$\boldsymbol{\Sigma} = -p_m \boldsymbol{I} + \boldsymbol{\Sigma}_s + \boldsymbol{\epsilon} \cdot (\boldsymbol{R} - \nabla \times \boldsymbol{V}), \tag{7.1}$$

in which $\boldsymbol{\epsilon}$ is the alternating 3-tensor. This result differs from available ones in two respects. In the first place, $\boldsymbol{V}$ represents a new contribution to the antisymmetric part of the stress for which we find the explicit result

$$\boldsymbol{V} = \mu_V(\overline{\boldsymbol{w}} - \boldsymbol{u}_m) \tag{7.2}$$

in which $\overline{\boldsymbol{w}}$ is the mean particle velocity and $\boldsymbol{u}_m$ the mixture volumetric flux. The parameter μ_V, which may be referred to as the *polar vortex viscosity*, is given by

$$\mu_V = \frac{\beta(3 - \beta\zeta)}{10H}\mu \tag{7.3}$$

in which $H = H(\beta)$ is the hindrance function for sedimentation and the coefficient $\zeta = \zeta(\beta)$ is defined in (4.6) and numerically computed with the result shown in

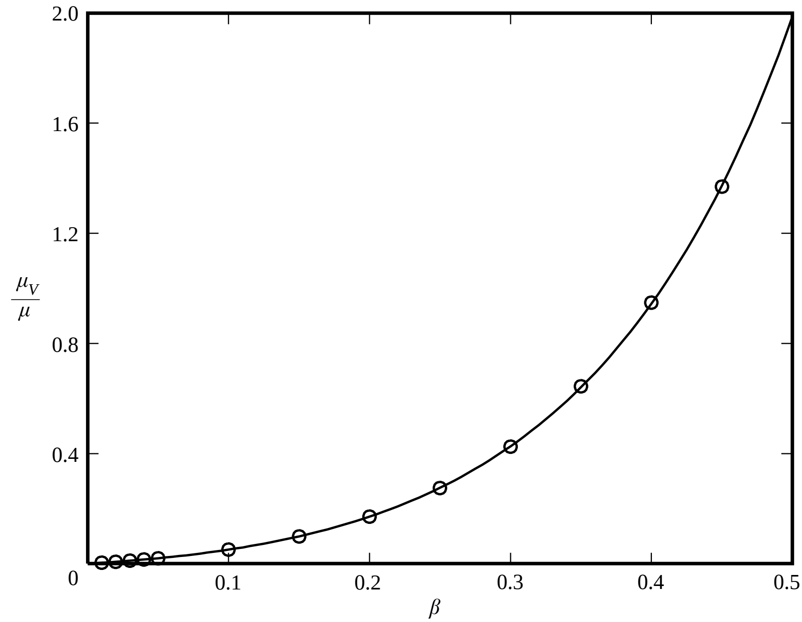

FIGURE 4. Dependence of the polar vortex viscosity defined in (7.3) on the particle volume fraction β. The solid line is obtained by using the result (5.2) for ζ and $H = (1-\beta)^{6.55-3.34\beta}$ for the hindrance function.

figure 2 and in table 1; μ_V is a monotonically increasing function of the particle volume fraction β. A graph of μ_V/μ vs. β is shown in figure 4.

An antisymmetric component of the stress tensor contributes to the rate of increase of the intrinsic angular momentum of the mixture (see e.g. Aris 1962). The physical mechanism by which the new term $\boldsymbol{V}$ identified in this study produces this effect is explained in § 6 and in figure 3. In essence, as is clear from the definition (3.16), it is related to an unbalanced tangential traction on the particle surface which introduces a couple density on the surface of a volume element of the suspension.

The other terms in (7.1) conform with Batchelor's results. p_m is the isotropic part of the stress which is shown in detail in (4.14). $\boldsymbol{\Sigma}_s$ is the symmetric stress including, among others, the contribution of the standard effective viscosity μ_{eff}; this term is studied in detail in Zhang & Prosperetti (2006). $\boldsymbol{R}$ is the antisymmetric component of the stress found by Batchelor (1970) and given by

$$\boldsymbol{R} = \mu_R(2\overline{\boldsymbol{\Omega}} - \nabla \times \boldsymbol{u}_m), \tag{7.4}$$

in which $\overline{\boldsymbol{\Omega}}$ is the mean particle angular velocity and the *axial vortex viscosity* μ_R is given by $\mu_R/\mu = 3\beta/(2R)$ with $R = R(\beta)$ the hindrance function for rotation. Unlike the $\boldsymbol{V}$ contribution, this one vanishes when there are no external couples acting on uniformly distributed particles.

With the previous expression (7.1) for $\boldsymbol{\Sigma}$, when the external force acting on the system reduces to gravity, the momentum equation for the suspension is

$$\nabla \cdot \boldsymbol{\Sigma} = -\left[(1-\beta)\rho_F + \beta\rho_P\right]\boldsymbol{g}. \tag{7.5}$$

An interesting situation to consider is that of a porous medium, for which $\overline{\boldsymbol{w}}$ vanishes identically so that, if the particle volume fraction is uniform,

$$-\nabla \times \nabla \times \boldsymbol{V} = \mu_V \nabla^2 \boldsymbol{u}_m \tag{7.6}$$

which is a contribution with the same form as that of the effective viscosity μ_{eff}. For $\beta = 20\,\%$, $30\,\%$, and $40\,\%$ we find $\mu_V/\mu_{eff} = 0.09$, 0.16, and 0.22 (the values of μ_{eff} used for this estimate are from Ladd 1990), which shows that the importance of our new term $\boldsymbol{V}$ increases with β and is not necessarily negligible.

This study was supported by the US Department of Energy under grant DE-FG02-99ER14966 and by the National Science Foundation under grant CTS-0210044.

Appendix A. Multipoles

In this appendix we show the relations between the force multipoles $T_{ij\ldots}$ and $U_{ij\ldots}$ that were used to find the relations quoted in §§4 and 5.

The Stokes flow field $(\boldsymbol{u}, p)$ in the neighbourhood of a particle can be represented as the sum of a part $(\boldsymbol{u}^r, p^r)$ regular and a part $(\boldsymbol{u}^s, p^s)$ singular at the particle centre. Lamb (1932; see also Kim & Karrila 1991) showed that these two components can be written as

$$\boldsymbol{u}^r = \frac{\nu}{a}\sum_{n=1}^{\infty}\left[\frac{1}{(n+1)(2n+3)}\left(\tfrac{1}{2}(n+3)s^2\nabla_s p_n - n\boldsymbol{s}p_n\right) + \nabla_s\phi_n + \nabla_s \times (\boldsymbol{s}\chi_n)\right], \tag{A 1}$$

and

$$\boldsymbol{u}^s = \frac{\nu}{a}\sum_{n=1}^{\infty}\left[\frac{1}{n(2n-1)}\left(-\tfrac{1}{2}(n-2)s^2\nabla_s p_{-n-1} + (n+1)\boldsymbol{s}p_{-n-1}\right)\right.$$
$$\left. + \nabla_s\phi_{-n-1} + \nabla_s \times (\boldsymbol{s}\chi_{-n-1})\right]. \tag{A 2}$$

In these equations p_n, ϕ_n, χ_n are dimensionless harmonics of positive order n of the dimensionless variable $\boldsymbol{s} = \boldsymbol{r}/a$, and $p_{-n-1}, \phi_{-n-1}, \chi_{-n-1}$ are dimensionless harmonics of negative order $-n-1$. The regular and singular harmonics are readily connected to each other by using the no-slip condition on the surface of the particle, assumed to translate with a velocity $\boldsymbol{w}$ and rotate with an angular velocity $\boldsymbol{\Omega}$:

$$s^{2n+1}p_{-n-1} = -\frac{n(2n-1)}{2(n+1)}p_n - \frac{n(4n^2-1)}{n+1}\phi_n + \frac{3}{4}\frac{a\boldsymbol{w}}{\nu}\cdot\boldsymbol{s}\delta_{n1}, \tag{A 3}$$

$$s^{2n+1}\phi^*_{-n-1} = -\frac{n}{(n+1)(2n+1)(2n+3)}p_n, \qquad s^{2n+1}\chi_{-n-1} = -\chi_n + \frac{a^2\boldsymbol{\Omega}}{\nu}\cdot\boldsymbol{s}\delta_{n1} \tag{A 4}$$

in which, following Mo & Sangani (1994), we have set $\phi^*_{-n-1} = \phi_{-n-1} - \frac{1}{2}p_{-n-1}/(2n+1)$.

It is readily deduced from these expressions that

$$\nabla^2\boldsymbol{u}^r|_{r=0} = \frac{\nu}{a}\nabla p_1|_{r=0} = -30\nabla(s^3\phi^*_{-2})|_{s=0}, \tag{A 5}$$

where here and in the following the derivatives of the potentials are with respect to the dimensionless variable $\boldsymbol{s}$.

It is shown in Tanksley & Prosperetti (2001), and can also be proven directly, that

$$\boldsymbol{T} = -4\pi\mu\nu\overline{\nabla(s^3p_{-2})}, \qquad \boldsymbol{U} = 40\pi\mu\nu\overline{\nabla(s^3\phi^*_{-2})} - \tfrac{4}{3}\pi\mu\nu\overline{\nabla(s^3p_{-2})}. \tag{A 6}$$

Here and in the following, evaluation of the derivatives of the potentials at $s = 0$, as in (A 5), is understood.

The second term of Batchelor's antisymmetric stress (3.15) is found to be

$$-\frac{a^2}{6}\partial_k\left[n\left(\epsilon_{im\ell}T_{\ell mk}+\epsilon_{km\ell}T_{\ell mi}\right)\right]=\partial_k\left[-\mu\nu\frac{v}{a}n\overline{\partial_i\partial_k(s^5\chi_{-3})}\right]=\partial_k\left(\mu\nu\frac{v}{a}n\overline{\partial_i\partial_k\chi_2}\right)$$
$$=\partial_k\left[\tfrac{1}{6}\mu\beta a^2\overline{\left(\partial_k\omega_i^r+\partial_i\omega_k^r\right)}\right] \qquad \text{(A 7)}$$

where $\boldsymbol{\omega}^r$ is the vorticity of the regular field. The symmetrization (3.10) was motivated by the fact that, without it, this term would have contained a mixture of the pseudo-scalars χ and scalars ϕ and p. In order to calculate this term we note that, for a uniform suspension with particles subjected to an equal force, we must have

$$a^2\overline{\partial_k\partial_j u_i^r}=A(\beta)\delta_{jk}W_i^0+B(\beta)\left(\delta_{ij}W_k^0+\delta_{ik}W_j^0\right). \qquad \text{(A 8)}$$

This relation holds also for particles subject to a couple as it is impossible to make a 3-tensor with the proper parity linear in the angular velocity. Upon taking the trace over the indices i and j, the result must vanish due to the incompressibility of the fluid, while the trace over j and k must reproduce (4.6). These two conditions are sufficient to conclude that

$$A=\tfrac{2}{5}\beta\zeta, \qquad B=-\tfrac{1}{10}\beta\zeta. \qquad \text{(A 9)}$$

With these results, it is easy to show that $\overline{\partial_k\omega_i^r+\partial_i\omega_k^r}=0$ so that the second term of Batchelor's antisymmetric stress (A 7) vanishes. For a non-uniform system, other terms such as $W_i^0\partial_j\partial_k\beta$ in principle could appear in the right-hand side of (A 8). However, for dimensional reasons, each derivative would have to be multiplied by a factor of a so that terms of this type would vanish in the continuum limit $a/L \to 0$.

The second term of the symmetric stress (3.14) is found to be

$$-\frac{a^2}{2}\partial_k(n\hat{T}_{ijk})=\mu\partial_k\left[-\frac{\beta}{48}\overline{\left(a^2\partial_k\partial_j(14+a^2\nabla^2)-\tfrac{14}{5}\delta_{jk}a^2\nabla^2\right)u_i^r}+\cdots\right] \qquad \text{(A 10)}$$

where the dots stand for two other terms obtained by cyclic permutation of i, j, k. A relation similar to (A 8) can be written for $a^4\overline{\partial_k\partial_j\nabla^2u_i^r}$, but now the traces over both i and j and i and k must vanish, and therefore the average of this quantity is zero. What remains can be evaluated using (A 8) to find $\hat{T}_{ijk}=0$.

The first term in the symmetric stress (4.17), in terms of the potentials, is

$$\hat{T}_{ij}=-\tfrac{2}{3}\pi\mu\nu\overline{\partial_i\partial_j(s^5p_{-3})}=\tfrac{2}{3}\pi\mu\nu\overline{\partial_i\partial_j\left(p_2+10\phi_2\right)}. \qquad \text{(A 11)}$$

By using (A 1), this expression can be related to the regular field evaluated at the particle centre to find (4.16).

For the first term of Batchelor's antisymmetric stress (3.15) we similarly have

$$\frac{a}{2}\epsilon_{ik\ell}nT_{k\ell}=-\frac{1}{2}n\overline{\int_{r=a}\mathrm{d}S_r\left[\boldsymbol{r}\times(\boldsymbol{\sigma}\cdot\boldsymbol{n})\right]_i}=-4\pi\mu\nu an\overline{\partial_i(s^3\chi_{-2})}=3n\nu\mu\left(\frac{\nu}{a^2}\overline{\partial_k\chi_1}-\overline{\boldsymbol{\Omega}}\right) \qquad \text{(A 12)}$$

from which (4.8) readily follows.

Appendix B. Calculation of $a^2\overline{\nabla^2\boldsymbol{u}^r}$

In his 1972 paper, by using renormalization, Batchelor proves the relation

$$a^2\overline{\nabla^2\boldsymbol{u}^B}=3\beta\boldsymbol{W}^0+o(\beta), \qquad \text{(B 1)}$$

where $\boldsymbol{u}^B$ is the velocity field at the centre of the test sphere evaluated in the absence of the test sphere. $\boldsymbol{u}^B$ differs from our $\boldsymbol{u}^r$ in that the latter includes the effect of the test sphere on the other spheres. However, if we set

$$a^2\overline{\nabla^2\boldsymbol{u}^r} = a^2\overline{\nabla^2\boldsymbol{u}^B} + a^2\overline{\nabla^2(\boldsymbol{u}^r - \boldsymbol{u}^B)}, \tag{B 2}$$

the second term decays sufficiently fast with distance from the centre of the test sphere for the ensemble average to be converted into a volume average. Following Batchelor's argument, for small concentrations it is sufficient to consider the effect of only one other sphere so that

$$a^2\overline{\nabla^2(\boldsymbol{u}^r - \boldsymbol{u}^B)} = a^2 n \int_{|\boldsymbol{x}-\boldsymbol{y}|\geqslant 2a} \nabla^2(\boldsymbol{u}^r - \boldsymbol{u}^B)\mathrm{d}\boldsymbol{y} = \frac{a^2 n}{\mu}\int_{|\boldsymbol{x}-\boldsymbol{y}|\geqslant 2a} \nabla_x(p^r - p^B)\mathrm{d}\boldsymbol{y}. \tag{B 3}$$

Since both p^r and p^B depend on $\boldsymbol{x} - \boldsymbol{y}$, we can replace ∇_x by $-\nabla_y$ and use the divergence theorem to find

$$\int_{|\boldsymbol{x}-\boldsymbol{y}|\geqslant 2a} \nabla_x(p^r - p^B)\mathrm{d}\boldsymbol{y} = \frac{1}{\mu}\int_{|\boldsymbol{r}|=2a} \boldsymbol{n}[p^r(\boldsymbol{r}) - p^B(\boldsymbol{r})]\,\mathrm{d}S_r \tag{B 4}$$

where $\boldsymbol{n}$ is the unit normal directed out of the sphere $|\boldsymbol{y} - \boldsymbol{x}| = 2a$. The contribution of the second term can be readily calculated using Stokes's solution to find

$$-\frac{a^2}{\mu}\int_{|\boldsymbol{r}|=2a} \boldsymbol{n}p^B(\boldsymbol{r})\,\mathrm{d}S_r = \frac{3}{2}\beta\boldsymbol{W}^0. \tag{B 5}$$

Let us now consider the contribution p^r. This is the pressure at $\boldsymbol{x}$ generated by a sphere centred at $\boldsymbol{y}$ when both spheres are falling together at the same velocity, $\boldsymbol{U}_1 = \boldsymbol{U}_2 = \boldsymbol{W}$, as they must by reversibility of Stokes flow if the force on them is the same. By equations (1), (3), and (4) in Jeffrey *et al.* (1993) and the notation of that paper, p^r is expressed

$$\frac{a}{\mu}p^r = \left(X^P_{11} + X^P_{12}\right)\boldsymbol{W}\cdot\boldsymbol{n}. \tag{B 6}$$

The common velocity of fall can be calculated from equation (1.10) of Jeffrey & Onishi (1984) with the result

$$\boldsymbol{n}\cdot\boldsymbol{W} = \left(x^a_{11} + x^a_{12}\right)\boldsymbol{n}\cdot\boldsymbol{W}^0 \tag{B 7}$$

so that

$$\frac{a}{\mu}p^r = \left(X^P_{11} + X^P_{12}\right)\left(x^a_{11} + x^a_{12}\right)\boldsymbol{n}\cdot\boldsymbol{W}^0. \tag{B 8}$$

Thus

$$\frac{a^2 n}{\mu}\int_{|\boldsymbol{y}-\boldsymbol{x}|=2a} \boldsymbol{n}p^r\,\mathrm{d}S_y = 4\beta\left[\left(X^P_{11} + X^P_{12}\right)\left(x^a_{11} + x^a_{12}\right)\right]_{r=2a}\boldsymbol{W}^0. \tag{B 9}$$

According to the results in table 1 of Jeffrey *et al.* (1993), at contact, $X^P_{11} + X^P_{12} \simeq -0.0118 - 0.1435 = -0.1553$ while, from tables 9 and 10 of Jeffrey & Onishi, $x^a_{11} + x^a_{12} \simeq 0.7550 + 0.7550 = 1.510$. Thus, upon combining this result with (B 5),

$$a^2\overline{\nabla^2(\boldsymbol{u}^r - \boldsymbol{u}^B)} \simeq 0.5371\beta\boldsymbol{W}^0 \tag{B 10}$$

from which, adding (B 1), the result (4.6) given in the text follows.

Appendix C. The hindrance function for rotation

We adopt a coordinate frame rotating with the angular velocity of the volumetric flux $\boldsymbol{u}_m$. In this frame, define the angular velocity of an isolated particle subject to a couple $\boldsymbol{L}$ by

$$\boldsymbol{\Omega}^0 = \frac{\boldsymbol{L}}{8\pi\mu a^3}. \tag{C 1}$$

From Faxén's theorem for rotation, or directly from (A 1) and (A 2), we have

$$\boldsymbol{\Omega}^0 = \overline{\boldsymbol{\Omega}} - \tfrac{1}{2}\overline{\nabla \times \boldsymbol{u}^r} = \overline{\boldsymbol{\Omega}} - \tfrac{1}{2}\overline{\boldsymbol{\omega}^r}. \tag{C 2}$$

Similarly to (B 2), we write

$$\overline{\boldsymbol{\Omega}} = \boldsymbol{\Omega}^0 + \tfrac{1}{2}\overline{\boldsymbol{\omega}^B} + \tfrac{1}{2}\overline{(\boldsymbol{\omega}^r - \boldsymbol{\omega}^B)} \tag{C 3}$$

where $\boldsymbol{\omega}^r$ is the vorticity of the regular field at the centre $\boldsymbol{x}$ of the test particle and $\boldsymbol{\omega}^B$ is the vorticity at $\boldsymbol{x}$ in the absence of the test particle. This decomposition is analogous to equation (3.3) in Batchelor (1972). We now follow his renormalization procedure noting that, since the unconditional average of $\boldsymbol{\omega}^B$ vanishes due to uniformity, we have

$$\overline{\boldsymbol{\omega}^B}(\boldsymbol{x}) = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N \boldsymbol{\omega}^B(\boldsymbol{x}, \mathscr{C}^N)[P(\mathscr{C}^N|\boldsymbol{x}) - P(\mathscr{C}^N)]. \tag{C 4}$$

At this point, following the same steps leading to equation (4.6) of Batchelor (1972), we find that

$$\overline{\boldsymbol{\omega}^B}(\boldsymbol{x}) = -n\int_{r\leqslant 2a} \boldsymbol{\omega}^B(\boldsymbol{x}+\boldsymbol{r}, \boldsymbol{x})\,\mathrm{d}\boldsymbol{r}, \tag{C 5}$$

where, for $r < a$, $\boldsymbol{\omega}^B$ equals $2\boldsymbol{\Omega}^0$, the vorticity inside a particle, while, for $a < r < 2a$, it is the vorticity in the fluid given by

$$\boldsymbol{\omega} = \frac{a^3}{r^3}(3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} - \boldsymbol{I})\cdot\boldsymbol{\Omega}^0, \tag{C 6}$$

in which $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. The result of this calculation is

$$\overline{\boldsymbol{\omega}^B}(\boldsymbol{x}) = -\beta\boldsymbol{\Omega}^0. \tag{C 7}$$

For the second term of (C 3) we may write

$$\frac{1}{2}\overline{(\boldsymbol{\omega}^r - \boldsymbol{\omega}^B)} = \frac{1}{2}n\int_{r\geqslant 2a}(\boldsymbol{\omega}^r - \boldsymbol{\omega}^B)\,\mathrm{d}\boldsymbol{r} \tag{C 8}$$

as the integral is now unconditionally convergent at infinity thanks to the presence of the second term. From Faxén's theorem, $\frac{1}{2}\boldsymbol{\omega}^r$ equals the angular velocity $\boldsymbol{\Omega}$ of the test particle minus $\boldsymbol{\Omega}^0$ so that

$$\frac{1}{2}\overline{(\boldsymbol{\omega}^r - \boldsymbol{\omega}^B)} = n\int_{r\geqslant 2a}\left(\boldsymbol{\Omega} - \boldsymbol{\Omega}^0 - \frac{1}{2}\boldsymbol{\omega}^B\right)\mathrm{d}\boldsymbol{r}. \tag{C 9}$$

Jeffrey & Onishi (1984) in their equation (1.10) give

$$\boldsymbol{\Omega} = \left[\left(x_{11}^c + x_{12}^c\right)\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + \left(y_{11}^c + y_{12}^c\right)(\boldsymbol{I} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}})\right]\cdot\boldsymbol{\Omega}^0. \tag{C 10}$$

Thus

$$\frac{1}{2}\overline{(\boldsymbol{\omega}^r - \boldsymbol{\omega}^B)} = \frac{4}{3}\pi n\boldsymbol{\Omega}^0\int_{r=2a}^{\infty}\left[x_{11}^c + x_{12}^c + 2\left(y_{11}^c + y_{12}^c\right) - 3\right]r^2\,\mathrm{d}r. \tag{C 11}$$

The calculation of this integral requires a tabulation of the functions appearing in (C 10). The information given in Jeffrey & Onishi (1984) is not very detailed and, therefore, we can only carry out a calculation of limited accuracy with the result $-0.5\beta\boldsymbol{\Omega}^0$, where the numerical coefficient has a likely error of less than 2 %. Upon combining with (C 7), we then find (4.9).

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